Decentralized adaptation in interconnected uncertain systems with nonlinear parametrization

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Abstract

We propose a technique for the design and analysis of decentralized adaptation algorithms in interconnected dynamical systems. Our technique does not require Lyapunov stability of the target dynamics and allows nonlinearly parameterized uncertainties. We show that for the considered class of systems, conditions for reaching the control goals can be formulated in terms of the nonlinear L_2-gains of target dynamics of each interconnected subsystem. Equations for decentralized controllers and corresponding adaptation algorithms are also explicitly provided.

Keywords: nonlinear parametrization; unstable, non-equilibrium dynamics; decentralized adaptive control; monotone functions

Notation

According to the standard convention, $\mathbb{R}$ defines the field of real numbers and $\mathbb{R}_{\geq c} = \{x \in \mathbb{R} | x \geq c\}$, $\mathbb{R}_+ = \mathbb{R}_{\geq 0}$; symbol $\mathbb{R}^n$ stands for a linear space $\mathcal{L}(\mathbb{R})$ over the field of reals with $\dim\{\mathcal{L}(\mathbb{R})\} = n$; $\|\mathbf{x}\|$ denotes the Euclidian norm of $\mathbf{x} \in \mathbb{R}^n$; $\mathcal{C}^k$ denotes the space of functions that are at least k times differentiable; $\mathcal{K}$ denotes the class of all strictly increasing functions $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\kappa(0) = 0$. By $L_p^n[t_0, T]$, where $T > 0$, $p \geq 1$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\|\mathbf{f}\|_{p,[t_0,T]} = \left(\int_0^T \|\mathbf{f}(\tau)\|^p d\tau\right)^{1/p} < \infty$; $\|\mathbf{f}\|_{p,[t_0,T]}$ denotes the $L_p^n[t_0, T]$-norm of $\mathbf{f}(t)$. By $L_\infty^n[t_0, T]$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty,[t_0,T]} = \operatorname{ess\,sup}\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$, and $\|\mathbf{f}\|_{\infty,[t_0,T]}$ stands for the $L_\infty^n[t_0, T]$ norm of $\mathbf{f}(t)$.

A function $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be locally bounded if for any $\|\mathbf{x}\| < \delta$ there exists a constant $D(\delta) > 0$ such that the following inequality holds: $\|\mathbf{f}(\mathbf{x})\| \leq D(\delta)$. Let Γ be an $n \times n$ square matrix, then $\Gamma > 0$ denotes a positive definite (symmetric) matrix, and Γ^{-1} is the inverse of Γ. By $\Gamma \geq 0$ we denote a positive semi-definite matrix, $\|\mathbf{x}\|_\Gamma^2$ to denotes the quadratic form: $\mathbf{x}^T \Gamma \mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$. The notation $|\cdot|$ stands for the modulus of a scalar. The solution of a

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system of differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \boldsymbol{\theta}, \mathbf{u}),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{u} : \mathbb{R}_+ \rightarrow \mathbb{R}^m,\ \boldsymbol{\theta} \in \mathbb{R}^d$ for $t \geq t_0$ will be denoted as $\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, \mathbf{u})$, or simply as $\mathbf{x}(t)$ if it is clear from the context what the values of $\mathbf{x}_0, \boldsymbol{\theta}$ are and how the function $\mathbf{u}(t)$ is defined.

Let $\mathbf{u} : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ be a function of state $\mathbf{x}$, parameters $\hat{\boldsymbol{\theta}}$, and time t. Let in addition both $\mathbf{x}$ and $\hat{\boldsymbol{\theta}}$ be functions of t. Then in case the arguments of $\mathbf{u}$ are clearly defined by the context, we will simply write $\mathbf{u}(t)$ instead of $\mathbf{u}(\mathbf{x}(t), \hat{\boldsymbol{\theta}}(t), t)$.

The (forward complete) system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \boldsymbol{\theta}, \mathbf{u}(t))$, is said to have an $L_p^m[t_0, T] \mapsto L_q^n[t_0, T]$, gain ($T \geq t_0$, $p, q \in \mathbb{R}_{\geq 1} \cup \infty$) with respect to its input $\mathbf{u}(t)$ if and only if $\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, \mathbf{u}(t)) \in L_q^n[t_0, T]$ for any $\mathbf{u}(t) \in L_p^m[t_0, T]$ and there exists a function $\gamma_{q,p} : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following inequality holds: $\|\mathbf{x}(t)\|_{q,[t_0,T]} \leq \gamma_{q,p}(\mathbf{x}_0, \boldsymbol{\theta}, \|\mathbf{u}(t)\|_{p,[t_0,T]})$. The function $\gamma_{q,p}(\mathbf{x}_0, \boldsymbol{\theta}, \|\mathbf{u}(t)\|_{p,[t_0,T]})$ is assumed to be non-decreasing in $\|\mathbf{u}(t)\|_{p,[t_0,T]}$, and locally bounded in its arguments.

For notational convenience when dealing with vector fields and partial derivatives we will use the following extended notion of the Lie derivative of a function. Let $\mathbf{x} \in \mathbb{R}^n$ and assume $\mathbf{x}$ can be partitioned as follows $\mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2$, where $\mathbf{x}_1 \in \mathbb{R}^q$, $\mathbf{x}_1 = (x_{11}, \ldots, x_{1q})^T$, $\mathbf{x}_2 \in \mathbb{R}^p$, $\mathbf{x}_2 = (x_{21}, \ldots, x_{2p})^T$, $q + p = n$, and $\oplus$ denotes the concatenation of two vectors. Define $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{f}_1(\mathbf{x}) \oplus \mathbf{f}_2(\mathbf{x})$, where $\mathbf{f}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^q$, $\mathbf{f}_1(\cdot) = (f_{11}(\cdot), \ldots, f_{1q}(\cdot))^T$, $\mathbf{f}_2 : \mathbb{R}^n \rightarrow \mathbb{R}^p$, $\mathbf{f}_2(\cdot) = (f_{21}(\cdot), \ldots, f_{2p}(\cdot))^T$. Then $L_{\mathbf{f}_i(\mathbf{x})}\psi(\mathbf{x}, t)$, $i \in \{1, 2\}$ denotes the Lie derivative of the function $\psi(\mathbf{x}, t)$ with respect to the vector field $\mathbf{f}_i(\mathbf{x}, \boldsymbol{\theta})$: $L_{\mathbf{f}_i(\mathbf{x})}\psi(\mathbf{x}, t) = \sum_j^{\dim \mathbf{x}_i} \frac{\partial \psi(\mathbf{x}, t)}{\partial x_{ij}} f_{ij}(\mathbf{x}, \boldsymbol{\theta})$.

1 Introduction

We consider the problem how to control the behavior of complex dynamical systems composed of interconnected lower-dimensional subsystems. Centralized control of these systems is practically inefficient because of high demands for computational power, measurements and prohibitive communication cost. On the other hand, standard decentralized solutions often face severe limitations due to the deficiency of information about the interconnected subsystems. In addition, the nature of their their interconnections may vary depending on conditions in the environment. In order to address these problems in their most general setup, decentralized adaptive control is needed.

Currently there is a large literature on decentralized adaptive control which contains successful solutions to problems of adaptive stabilization [6, 8], tracking [7, 8, 17, 18], and output regulation [9, 23] of linear and nonlinear systems. In most of these cases the problem of decentralized control is solved within the conventional framework of adaptive stabilization/tracking/regulation by a family of linearly parameterized controllers. While these results may be successfully implemented in a large variety of technical and artificial systems, there is room for further improvements. In particular, when the target dynamics of the systems is not stable in the Lyapunov sense but intermittent, meta-stable, or multi-stable [1, 15, 19] or when the uncertainties are nonlinearly parameterized [2, 3, 4, 11], and no domination of the uncertainties by feedback is allowed.

In the present article we address these issues at once for a class of nonlinear dynamical sys-

tems. Our contribution is that we provide conditions ensuring forward-completeness, boundedness and asymptotic reaching of the goal for a pair of interconnected systems with uncertain coupling and parameters. Our method does not require availability of a Lyapunov function for the desired motions in each subsystem, nor linear parametrization of the controllers. Our results can straightforwardly be extended to interconnection of arbitrary many (but still, a finite number of) subsystems. Explicit equations for corresponding decentralized adaptive controllers are also provided.

The paper is organized as follows. In Section 2 we provide a formal statement of the problem, Section 3 contains necessary preliminaries and auxiliary results. In Section 4 we present the main results of our current contribution, and in Section 5 we provide concluding remarks to our approach.

2 Problem Formulation

Let us consider two interconnected systems $\mathcal{S}_x$ and $\mathcal{S}_y$:

$$\mathcal{S}_x: \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}_x) + \gamma_y(\mathbf{y}, t) + \mathbf{g}(\mathbf{x})u_x \tag{1}$$

$$\mathcal{S}_y: \quad \dot{\mathbf{y}} = \mathbf{q}(\mathbf{y}, \boldsymbol{\theta}_y) + \gamma_x(\mathbf{x}, t) + \mathbf{z}(\mathbf{y})u_y \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$, $\mathbf{y} \in \mathbb{R}^{n_y}$ are the state vectors of systems $\mathcal{S}_x$ and $\mathcal{S}_y$, vectors $\boldsymbol{\theta}_x \in \mathbb{R}^{n_{\theta x}}$, $\boldsymbol{\theta}_y \in \mathbb{R}^{n_{\theta y}}$ are unknown parameters, functions $\mathbf{f}: \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\theta x}} \rightarrow \mathbb{R}^{n_x}$, $\mathbf{q}: \mathbb{R}^{n_y} \times \mathbb{R}^{n_{\theta y}} \rightarrow \mathbb{R}^{n_y}$, $\mathbf{g}: \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$, $\mathbf{z}: \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$ are continuous and locally bounded. Functions $\gamma_y: \mathbb{R}^{n_y} \times \mathbb{R}_+ \rightarrow \mathbb{R}_n$, $\gamma_x: \mathbb{R}^{n_x} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n_y}$, stand for nonlinear, non-stationary and, in general, unknown couplings between systems $\mathcal{S}_x$ and $\mathcal{S}_y$, and $u_x \in \mathbb{R}$, $u_y \in \mathbb{R}$ are the control inputs.

In the present paper we are interested in the following problem

Problem 1 Let $\psi_x: \mathbb{R}^{n_x} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, $\psi_y: \mathbb{R}^{n_y} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be the goal functions for systems $\mathcal{S}_x$, $\mathcal{S}_y$ respectively. In the other words, for some values $\varepsilon_x \in \mathbb{R}_+$, $\varepsilon_y \in \mathbb{R}_+$ and time instant $t^* \in \mathbb{R}_+$, inequalities

$$\|\psi_x(\mathbf{x}(t), t)\|_{\infty,[t^*,\infty]} \leq \varepsilon_x, \ \|\psi_y(\mathbf{y}(t), t)\|_{\infty,[t^*,\infty]} \leq \varepsilon_y \tag{3}$$

specify the desired state of interconnection (1), (2). Derive functions $u_x(\mathbf{x}, t)$, $u_y(\mathbf{y}, t)$ such that for all $\boldsymbol{\theta}_x \in \mathbb{R}^{n_{\theta x}}$, $\boldsymbol{\theta}_y \in \mathbb{R}^{n_{\theta y}}$

1) interconnection (1), (2) is forward-complete;

2) the trajectories $\mathbf{x}(t)$, $\mathbf{y}(t)$ are bounded;

3) for given values of ε_x, ε_y, some $t^* \in \mathbb{R}_+$ exists such that inequalities (3) are satisfied or, possibly, both functions $\psi_x(\mathbf{x}(t), t)$, $\psi_y(\mathbf{y}(t), t)$ converge to zero as $t \rightarrow \infty$.

Function $u_x(\cdot)$ should not depend explicitly on $\mathbf{y}$ and, symmetrically, function $u_y(\cdot)$ should not depend explicitly on $\mathbf{x}$. The general structure of the desired configuration of the control scheme is provided in Figure 1.

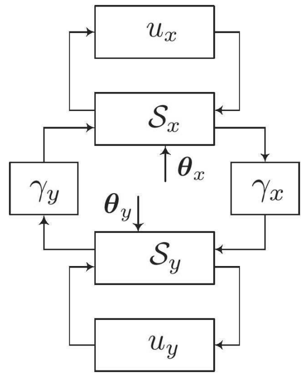

Figure 1: General structure of interconnection

In the next sections we provide sufficient conditions, ensuring solvability of Problem 1 and we also explicitly derive functions $u_x(\mathbf{x}, t)$ and $u_y(\mathbf{y}, t)$ which satisfy requirements 1) – 3) of Problem 1. We start with the introduction of a new class of adaptive control schemes and continue by providing the input-output characterizations of the controlled systems. These results are given in Section 3. Then, using these characterizations, in Section 4 we provide the main results of our study.

3 Assumptions and properties of the decoupled systems

Let the following system be given:

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{f}_1(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u,\\ \dot{\mathbf{x}}_2 =& \mathbf{f}_2(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{g}_2(\mathbf{x})u,\end{aligned} \tag{4}$$

where

$$\mathbf{x}_1 = (x_{11}, \ldots, x_{1q})^T \in \mathbb{R}^q;\ \mathbf{x}_2 = (x_{21}, \ldots, x_{2p})^T \in \mathbb{R}^p;$$
$$\mathbf{x} = (x_{11}, \ldots, x_{1q}, x_{21}, \ldots, x_{2p})^T \in \mathbb{R}^n$$

$\boldsymbol{\theta} \in \Omega_\theta \in \mathbb{R}^d$ is a vector of unknown parameters, and Ω_θ is a closed bounded subset of $\mathbb{R}^d$; $u \in \mathbb{R}$ is the control input, and functions $\mathbf{f}_1 : \mathbb{R}^n \to \mathbb{R}^q$, $\mathbf{f}_2 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^p$, $\mathbf{g}_1 : \mathbb{R}^n \to \mathbb{R}^q$, $\mathbf{g}_2 : \mathbb{R}^n \to \mathbb{R}^p$ are continuous and locally bounded. The vector $\mathbf{x} \in \mathbb{R}^n$ is the state vector, and vectors $\mathbf{x}_1$, $\mathbf{x}_2$ are referred to as *uncertainty-independent* and *uncertainty-dependent* partition of $\mathbf{x}$, respectively. For the sake of compactness we will also use the following description of (4):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}) + \mathbf{g}(\mathbf{x})u, \tag{5}$$

where

$$\mathbf{g}(\mathbf{x}) = (g_{11}(\mathbf{x}), \ldots, g_{1q}(\mathbf{x}), g_{21}(\mathbf{x}), \ldots, g_{2p}(\mathbf{x}))^T,$$
$$\mathbf{f}(\mathbf{x}) = (f_{11}(\mathbf{x}), \ldots, f_{1q}(\mathbf{x}), f_{21}(\mathbf{x}, \boldsymbol{\theta}), \ldots, f_{2p}(\mathbf{x}, \boldsymbol{\theta}))^T.$$

As a measure of closeness of trajectories $\mathbf{x}(t)$ to the desired state we introduce the error or goal function $\psi : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R},\ \psi \in \mathcal{C}^1$. We suppose also that for the chosen function $\psi(\mathbf{x}, t)$ satisfies the following:

Assumption 1 (Target operator) *For the given function $\psi(\mathbf{x}, t) \in \mathcal{C}^1$ the following property holds:*

$$\|\mathbf{x}(t)\|_{\infty,[t_0,T]} \leq \tilde{\gamma}\left(\mathbf{x}_0, \boldsymbol{\theta}, \|\psi(\mathbf{x}(t), t)\|_{\infty,[t_0,T]}\right) \tag{6}$$

where $\tilde{\gamma}\left(\mathbf{x}_0, \boldsymbol{\theta}, \|\psi(\mathbf{x}(t), t)\|_{\infty,[t_0,T]}\right)$ is a locally bounded and non-negative function of its arguments.

Assumption 1 can be interpreted as a sort of *unboundedness observability* property [10] of system (4) with respect to the "output" function $\psi(\mathbf{x}, t)$. It can also be viewed as a *bounded input - bounded state* assumption for system (4) along the constraint $\psi(\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, u(\mathbf{x}(t), t)), t) = \upsilon(t)$, where the signal $\upsilon(t)$ serves as a new input. If, however, boundedness of the state is not explicitly required (i.e. it is guaranteed by additional control or follows from the physical properties of the system itself), Assumption 1 can be removed from the statements of our results.

Let us specify a class of control inputs u which can ensure boundedness of $\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, u)$ for every $\boldsymbol{\theta} \in \Omega_\theta$ and $\mathbf{x}_0 \in \mathbb{R}^n$. According to (6), boundedness of $\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, u)$ is ensured if we find a control input u such that $\psi(\mathbf{x}(t), t) \in L^1_\infty[t_0, \infty]$. For this objective consider the dynamics of system (5) with respect to $\psi(\mathbf{x}, t)$:

$$\dot{\psi} = L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})}\psi(\mathbf{x}, t) + L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x}, t)u + \frac{\partial\psi(\mathbf{x}, t)}{\partial t}, \tag{7}$$

Assuming that the inverse $\left(L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x}, t)\right)^{-1}$ exists everywhere, we may choose the control input u in the following class of functions:

$$\begin{aligned} u(\mathbf{x}, \hat{\boldsymbol{\theta}}, \boldsymbol{\omega}, t) &= \frac{1}{L_{\mathbf{g}(\mathbf{x})}\psi(\mathbf{x}, t)}\left(-L_{\mathbf{f}(\mathbf{x},\hat{\boldsymbol{\theta}})}\psi(\mathbf{x}, t) - \varphi(\psi, \boldsymbol{\omega}, t) - \frac{\partial\psi(\mathbf{x}, t)}{\partial t}\right) \\ &\varphi :\ \mathbb{R} \times \mathbb{R}^w \times \mathbb{R}_+ \rightarrow \mathbb{R} \end{aligned} \tag{8}$$

where $\boldsymbol{\omega} \in \Omega_\omega \subset \mathbb{R}^w$ is a vector of *known* parameters of the function $\varphi(\psi, \boldsymbol{\omega}, t)$. Denoting $L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})}\psi(\mathbf{x}, t) = f(\mathbf{x}, \boldsymbol{\theta}, t)$ and taking into account (8) we may rewrite equation (7) in the following manner:

$$\dot{\psi} = f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - \varphi(\psi, \boldsymbol{\omega}, t) \tag{9}$$

For the purpose of the present article, instead of (9) it is worthwhile to consider the extended equation:

$$\dot{\psi} = f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - \varphi(\psi, \boldsymbol{\omega}, t) + \varepsilon(t), \tag{10}$$

where, if not stated overwise, the function $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\varepsilon \in L^1_2[t_0, \infty] \cap C^0$. One of the immediate advantages of equation (10) in comparison with (9) is that it allows us to take the presence of coupling between interconnected systems into consideration.

Let us now specify the desired properties of the function $\varphi(\psi, \boldsymbol{\omega}, t)$ in (8), (10). The majority of known algorithms for parameter estimation and adaptive control [12, 13, 14, 16] assume global (Lyapunov) stability of system (10) for $\boldsymbol{\theta} \equiv \hat{\boldsymbol{\theta}}$. In our study, however, we refrain from this standard, restrictive requirement. Instead we propose that finite energy of the signal $f(\mathbf{x}(t), \boldsymbol{\theta}, t) - f(\mathbf{x}(t), \hat{\boldsymbol{\theta}}(t), t)$, defined for example by its $L_2^1[t_0, \infty]$ norm with respect to the variable t, results in finite deviation from the target set given by the equality $\psi(\mathbf{x}, t) = 0$. Formally this requirement is introduced in Assumption 2:

Assumption 2 (Target dynamics operator) *Consider the following system:*

$$\dot{\psi} = -\varphi(\psi, \boldsymbol{\omega}, t) + \zeta(t), \tag{11}$$

where $\zeta : \mathbb{R}_+ \to \mathbb{R}$ and $\varphi(\psi, \boldsymbol{\omega}, t)$ is defined in (10). Then for every $\boldsymbol{\omega} \in \Omega_\omega$ system (11) has $L_2^1[t_0, \infty] \mapsto L_\infty^1[t_0, \infty]$ gain with respect to input $\zeta(t)$. In other words, there exists a function $\gamma_{\infty,2}$ such that

$$\|\psi(t)\|_{\infty,[t_0,T]} \leq \gamma_{\infty,2}(\psi_0, \boldsymbol{\omega}, \|\zeta(t)\|_{2,[t_0,T]}), \quad \forall\ \zeta(t) \in L_2^1[t_0, T] \tag{12}$$

In contrast to conventional approaches, Assumption 2 does not require global *asymptotic stability* of the origin of the unperturbed (i.e for $\zeta(t) = 0$) system (11). When the stability of the target dynamics $\dot{\psi} = -\varphi(\psi, \boldsymbol{\omega}, t)$ is known a-priori, one of the benefits of Assumption 2 is that there is no need to know a *particular Lyapunov function* of the unperturbed system.

So far we have introduced basic assumptions on system (4) and the class of feedback considered in this article. Let us now specify the class of functions $f(\mathbf{x}, \boldsymbol{\theta}, t)$ in (10). Since general parametrization of function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ is methodologically difficult to deal with, but solutions provided for nonlinearities with convenient linear re-parametrization often yield physically implausible models and large number of unknown parameters, we have opted for a new class of parameterizations. As a candidate for such a parametrization we suggest nonlinear functions that satisfy the following assumption:

Assumption 3 (Monotonicity and Growth Rate in Parameters) *For the given function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ in (10) there exists function $\boldsymbol{\alpha}(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^d$, $\boldsymbol{\alpha}(\mathbf{x}, t) \in \mathcal{C}^1$ and positive constant $D > 0$ such that*

$$(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t))(\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})) \geq 0 \tag{13}$$

$$|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| \leq D|\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})| \tag{14}$$

This set of conditions naturally extends from systems that are linear in parameters to those with nonlinear parametrization. Examples and models of physical and artificial systems which satisfy Assumption 3 (at least for bounded $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}} \in \Omega_\theta$) can be found in the following references [2, 3, 4, 5, 11]. Assumption 3 bounds the growth rate of the difference $|f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)|$ by the functional $D|\boldsymbol{\alpha}(\mathbf{x}, t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})|$. In addition, it might also be useful to have an estimate of $|f(\mathbf{x}, \boldsymbol{\theta}, t) - f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)|$ from below, as specified in Assumption 4:

Assumption 4 *For the given function $f(\mathbf{x},\boldsymbol{\theta},t)$ in (10) and function $\boldsymbol{\alpha}(\mathbf{x},t)$, satisfying Assumption 3, there exists a positive constant $D_1 > 0$ such that*

$$|f(\mathbf{x},\hat{\boldsymbol{\theta}},t) - f(\mathbf{x},\boldsymbol{\theta},t)| \geq D_1|\boldsymbol{\alpha}(\mathbf{x},t)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})| \tag{15}$$

In problems of adaptation, parameter and optimization estimation, effectiveness of the algorithms often depends on how "good" the nonlinearity $f(\mathbf{x},\boldsymbol{\theta},t)$ is, and how predictable is the system's behavior. As a measure of goodness and predictability usually the substitutes as smoothness and boundedness are considered. In our study, we distinguish several of such specific properties of the functions $f(\mathbf{x},\boldsymbol{\theta},t)$ and $\varphi(\psi,\boldsymbol{\omega},t)$. These properties are provided below.

H 1 *The function $f(\mathbf{x},\boldsymbol{\theta},t)$ is locally bounded with respect to $\mathbf{x}$, $\boldsymbol{\theta}$ uniformly in t.*

H 2 *The function $f(\mathbf{x},\boldsymbol{\theta},t) \in \mathcal{C}^1$, and $\partial f(\mathbf{x},\boldsymbol{\theta},t)/\partial t$ is locally bounded with respect to $\mathbf{x}$, $\boldsymbol{\theta}$ uniformly in t.*

H 3 *The function $\varphi(\psi,\boldsymbol{\omega},t)$ is locally bounded in ψ, $\boldsymbol{\omega}$ uniformly in t.*

Let us show that under an additional structural requirement, which relates properties of the function $\boldsymbol{\alpha}(\mathbf{x},t)$ and vector-field $\mathbf{f}(\mathbf{x},\boldsymbol{\theta}) = \mathbf{f}_1(\mathbf{x},\boldsymbol{\theta}) \oplus \mathbf{f}_2(\mathbf{x},\boldsymbol{\theta})$ in (4), (5), there exist adaptive algorithms ensuring that the following desired property holds:

$$\mathbf{x}(t) \in L_\infty^n[t_0,\infty];\ f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x},\hat{\boldsymbol{\theta}}(t),t) \in L_2^1[t_0,\infty] \tag{16}$$

Consider the following adaptation algorithms:

$$\begin{aligned}
\hat{\boldsymbol{\theta}}(\mathbf{x},t) &= \Gamma(\hat{\boldsymbol{\theta}}_P(\mathbf{x},t) + \hat{\boldsymbol{\theta}}_I(t));\ \Gamma \in \mathbb{R}^{d\times d},\ \Gamma > 0 \\
\hat{\boldsymbol{\theta}}_P(\mathbf{x},t) &= \psi(\mathbf{x},t)\boldsymbol{\alpha}(\mathbf{x},t) - \Psi(\mathbf{x},t) \\
\dot{\hat{\boldsymbol{\theta}}}_I &= \varphi(\psi(\mathbf{x},t),\boldsymbol{\omega},t)\boldsymbol{\alpha}(\mathbf{x},t) + \mathcal{R}(\mathbf{x},\hat{\boldsymbol{\theta}},u(\mathbf{x},\hat{\boldsymbol{\theta}},t),t),
\end{aligned} \tag{17}$$

where the function $\mathcal{R}(\mathbf{x},\hat{\boldsymbol{\theta}},u(\mathbf{x},\hat{\boldsymbol{\theta}},t),t) : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^d$ in (17) is given as follows:

$$\begin{aligned}
&\mathcal{R}(\mathbf{x},u(\mathbf{x},\hat{\boldsymbol{\theta}},t),t) = \partial\Psi(\mathbf{x},t)/\partial t - \psi(\mathbf{x},t)(\partial\boldsymbol{\alpha}(\mathbf{x},t)/\partial t + L_{\mathbf{f}_1}\boldsymbol{\alpha}(\mathbf{x},t)) \\
&+ L_{\mathbf{f}_1}\Psi(\mathbf{x},t) - (\psi(\mathbf{x},t)L_{\mathbf{g}_1}\boldsymbol{\alpha}(\mathbf{x},t) - L_{\mathbf{g}_1}\Psi(\mathbf{x},t))u(\mathbf{x},\hat{\boldsymbol{\theta}},t)
\end{aligned} \tag{18}$$

and function $\Psi(\mathbf{x},t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_d$, $\Psi(\mathbf{x},t) \in \mathcal{C}^1$ satisfies Assumption 5.

Assumption 5 *There exists a function $\Psi(\mathbf{x},t)$ such that*

$$\frac{\partial\Psi(\mathbf{x},t)}{\partial\mathbf{x}_2} - \psi(\mathbf{x},t)\frac{\partial\boldsymbol{\alpha}(\mathbf{x},t)}{\partial\mathbf{x}_2} = 0 \tag{19}$$

Additional restrictions imposed by this assumption will be discussed in some details after we summarize the properties of system (4), (8), (17), (18) in the following theorem.

Theorem 1 (Properties of the decoupled systems) *Let system (4), (10), (17), (18) be given and Assumptions 3, 4, 5 be satisfied. Then the following properties hold*

P1) Let for the given initial conditions $\mathbf{x}(t_0)$, $\hat{\boldsymbol{\theta}}_I(t_0)$ *and parameters vector* $\boldsymbol{\theta}$, *interval* $[t_0, T^*]$ *be the (maximal) time-interval of existence of solutions of the closed loop system (4), (10), (17), (18). Then*

$$\|f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t))\|_{2,[t_0,T^*]} \leq D_f(\boldsymbol{\theta}, t_0, \Gamma, \|\varepsilon(t)\|_{2,[t_0,T^*]}); \tag{20}$$

$$D_f(\boldsymbol{\theta}, t_0, \Gamma, \|\varepsilon(t)\|_{2,[t_0,T^*]}) = \left(\frac{D}{2}\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t_0)\|^2_{\Gamma^{-1}}\right)^{0.5} + \frac{D}{D_1}\|\varepsilon(t)\|_{2,[t_0,T^*]}$$

$$\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)\|^2_{\Gamma^{-1}} \leq \|\hat{\boldsymbol{\theta}}(t_0) - \boldsymbol{\theta}\|^2_{\Gamma^{-1}} + \frac{D}{2D_1^2}\|\varepsilon(t)\|^2_{2,[t_0,T^*]}$$

In addition, if Assumptions 1 and 2 are satisfied then

P2) $\psi(\mathbf{x}(t),t) \in L^1_\infty[t_0,\infty]$, $\mathbf{x}(t) \in L^n_\infty[t_0,\infty]$ *and*

$$\|\psi(\mathbf{x}(t),t)\|_{\infty,[t_0,\infty]} \leq \gamma_{\infty,2}\left(\psi(\mathbf{x}_0,t_0), \boldsymbol{\omega}, \mathcal{D}\right) \tag{21}$$

$$\mathcal{D} = D_f(\boldsymbol{\theta}, t_0, \Gamma, \|\varepsilon(t)\|_{2,[t_0,\infty]}) + \|\varepsilon(t)\|_{2,[t_0,\infty]}$$

P3) if properties H1, H3 hold, and system (11) has $L^1_2[t_0,\infty] \mapsto L^1_p[t_0,\infty]$, $p > 1$ *gain with respect to input* $\zeta(t)$ *and output* ψ *then*

$$\varepsilon(t) \in L^1_2[t_0,\infty] \cap L^1_\infty[t_0,\infty] \Rightarrow \lim_{t\to\infty} \psi(\mathbf{x}(t),t) = 0 \tag{22}$$

If, in addition, property H2 holds, and the functions $\boldsymbol{\alpha}(\mathbf{x},t)$, $\partial\psi(\mathbf{x},t)/\partial t$ *are locally bounded with respect to* $\mathbf{x}$ *uniformly in* t, *then*

P4) the following holds

$$\lim_{t\to\infty} f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t) = 0 \tag{23}$$

The proof of Theorem 1 and subsequent results are given in Section 6.

Let us briefly comment on Assumption 5. Let $\boldsymbol{\alpha}(\mathbf{x},t) \in \mathcal{C}^2$, $\boldsymbol{\alpha}(\mathbf{x},t) = \mathrm{col}(\alpha_1(\mathbf{x},t), \ldots, \alpha_d(\mathbf{x},t))$, then necessary and sufficient conditions for existence of the function $\Psi(\mathbf{x},t)$ follow from the Poincaré lemma:

$$\frac{\partial}{\partial \mathbf{x}_2}\left(\psi(\mathbf{x},t)\frac{\partial \alpha_i(\mathbf{x},t)}{\partial \mathbf{x}_2}\right) = \left(\frac{\partial}{\partial \mathbf{x}_2}\left(\psi(\mathbf{x},t)\frac{\partial \alpha_i(\mathbf{x},t)}{\partial \mathbf{x}_2}\right)\right)^T \tag{24}$$

This relation, in the form of conditions of existence of the solutions for function $\Psi(\mathbf{x},t)$ in (19), takes into account structural properties of system (4), (10). Indeed, consider partial derivatives $\partial\alpha_i(\mathbf{x},t)/\partial\mathbf{x}_2$, $\partial\psi(\mathbf{x},t)/\partial\mathbf{x}_2$ with respect to the vector $\mathbf{x}_2 = (x_{21}, \ldots, x_{2p})^T$. Let

$$\frac{\partial\psi(\mathbf{x},t)}{\partial \mathbf{x}_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 \end{pmatrix}, \ \frac{\partial\alpha_i(\mathbf{x},t)}{\partial \mathbf{x}_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 \end{pmatrix} \tag{25}$$

where the symbol $*$ denotes a function of $\mathbf{x}$ and t. Then condition (25) guarantees that equality (24) (and, subsequently, Assumption 5) holds. In case $\partial\alpha(\mathbf{x}_1 \oplus \mathbf{x}_2, t)/\partial\mathbf{x}_2 = 0$, Assumption 5

holds for arbitrary $\psi(\mathbf{x},t) \in \mathcal{C}^1$. If $\psi(\mathbf{x},t)$, $\boldsymbol{\alpha}(\mathbf{x},t)$ depend on a single component of $\mathbf{x}_2$, for instance $x_{2k},\ k \in \{0,\dots,p\}$, then conditions (25) hold and the function $\Psi(\mathbf{x},t)$ can be derived explicitly by integration

$$\Psi(\mathbf{x},t) = \int \psi(\mathbf{x},t)\frac{\boldsymbol{\alpha}(\mathbf{x},t)}{\partial x_{2k}} dx_{2k} \tag{26}$$

In all other cases, existence of the required function $\Psi(\mathbf{x},t)$ follows from (24).

In the general case, when $\dim\{\mathbf{x}_2\} > 1$, the problems of finding a function $\Psi(\mathbf{x},t)$ satisfying condition (19) can be avoided (or converted into one with an already known solutions such as (24), (26)) by the *embedding* technique proposed in [20]. The main idea of the method is to introduce an auxiliary system that is forward-complete with respect to input $\mathbf{x}(t)$

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \mathbf{f}_{\boldsymbol{\xi}}(\mathbf{x},\boldsymbol{\xi},t),\ \boldsymbol{\xi} \in \mathbb{R}^z \\ \mathbf{h}_\xi &= \mathbf{h}_\xi(\boldsymbol{\xi},t),\ \mathbb{R}^z \times \mathbb{R}_+ \to \mathbb{R}^h\end{aligned} \tag{27}$$

such that

$$\|f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}_1(t) \oplus \mathbf{h}_\xi(t) \oplus \mathbf{x}_2'(t),\boldsymbol{\theta},t)\|_{2,[t_0,T]} \leq C_\xi \in \mathbb{R}_+ \tag{28}$$

for all $T \geq t_0$, and $\dim\{\mathbf{h}_\xi\} + \dim\{\mathbf{x}_2'\} = p$. Then (10) can be rewritten as follows:

$$\dot{\psi} = f(\mathbf{x}_1 \oplus \mathbf{h}_\xi \oplus \mathbf{x}_2', \boldsymbol{\theta}, t) - f(\mathbf{x}_1 \oplus \mathbf{h}_\xi \oplus \mathbf{x}_2', \hat{\boldsymbol{\theta}}, t) - \varphi(\psi, \boldsymbol{\omega}, t) + \varepsilon_\xi(t), \tag{29}$$

where $\varepsilon_\xi(t) \in L_2^1[t_0,\infty]$, and $\dim\{\mathbf{x}_2'\} = p - h < p$. In principle, the dimension of $\mathbf{x}_2'$ could be reduced to 1 or 0. As soon as this is ensured, Assumption 5 will be satisfied and the results of Theorem 1 follow. Sufficient conditions ensuring the existence of such an embedding in the general case are provided in [20]. For systems in which the parametric uncertainty can be reduced to vector fields with low-triangular structure the embedding is given in [21].

4 Main Results

Without loss of generality let us rewrite interconnection (1), (2) as follows :

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u_x \\ \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x},\boldsymbol{\theta}_x) + \gamma_y(\mathbf{y},t) + \mathbf{g}_2(\mathbf{x})u_x\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{\mathbf{y}}_1 &= \mathbf{q}_1(\mathbf{y}) + \mathbf{z}_1(\mathbf{y})u_y \\ \dot{\mathbf{y}}_2 &= \mathbf{q}_2(\mathbf{y},\boldsymbol{\theta}_y) + \gamma_x(\mathbf{x},t) + \mathbf{z}_2(\mathbf{y})u_y\end{aligned} \tag{31}$$

Let us now consider the following control functions

$$\begin{aligned}u_x(\mathbf{x},\hat{\boldsymbol{\theta}}_x,\boldsymbol{\omega}_x,t) = (L_{\mathbf{g}(\mathbf{x})}\psi_x(\mathbf{x},t))^{-1}\Big(&-L_{\mathbf{f}(\mathbf{x},\hat{\boldsymbol{\theta}}_x)}\psi_x(\mathbf{x},t) - \varphi_x(\psi_x,\boldsymbol{\omega}_x,t) \\ &-\frac{\partial\psi_x(\mathbf{x},t)}{\partial t}\Big),\quad \varphi_x:\ \mathbb{R}\times\mathbb{R}^w\times\mathbb{R}_+ \to \mathbb{R}\end{aligned} \tag{32}$$

$$u_y(\mathbf{y},\hat{\boldsymbol{\theta}}_y,\boldsymbol{\omega}_y,t)=(L_{\mathbf{z}(\mathbf{y})}\psi_y(\mathbf{y},t))^{-1}\Big(-L_{\mathbf{q}(\mathbf{y},\hat{\boldsymbol{\theta}}_y)}\psi_y(\mathbf{y},t)-\varphi_y(\psi_y,\boldsymbol{\omega}_y,t)$$
$$-\frac{\partial\psi_y(\mathbf{y},t)}{\partial t}\Big),\quad \varphi_y:\ \mathbb{R}\times\mathbb{R}^w\times\mathbb{R}_+\to\mathbb{R} \tag{33}$$

These functions transform the original equations (30), (31) into the following form

$$\begin{aligned}\dot{\psi}_x&=-\varphi_x(\psi_x,\boldsymbol{\omega}_x,t)+f_x(\mathbf{x},\boldsymbol{\theta}_x,t)-f_x(\mathbf{x},\hat{\boldsymbol{\theta}}_x,t)+h_y(\mathbf{x},\mathbf{y},t)\\ \dot{\psi}_y&=-\varphi_y(\psi_x,\boldsymbol{\omega}_y,t)+f_y(\mathbf{y},\boldsymbol{\theta}_y,t)-f_y(\mathbf{y},\hat{\boldsymbol{\theta}}_y,t)+h_x(\mathbf{x},\mathbf{y},t),\end{aligned} \tag{34}$$

where

$$h_x(\mathbf{x},\mathbf{y},t)=L_{\gamma_y(\mathbf{y},t)}\psi_x(\mathbf{x},t),\ \ h_y(\mathbf{x},\mathbf{y},t)=L_{\gamma_x(\mathbf{x},t)}\psi_y(\mathbf{y},t)$$

$$f_x(\mathbf{x},\boldsymbol{\theta}_x,t)=L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta}_x)}\psi_x(\mathbf{x},t),\ \ f_y(\mathbf{x},\boldsymbol{\theta}_y,t)=L_{\mathbf{q}(\mathbf{y},\boldsymbol{\theta}_y)}\psi_y(\mathbf{y},t)$$

Consider the following adaptation algorithms

$$\begin{aligned}\hat{\boldsymbol{\theta}}_x(\mathbf{x},t)&=\Gamma_x(\hat{\boldsymbol{\theta}}_{P,x}(\mathbf{x},t)+\hat{\boldsymbol{\theta}}_{I,x}(t));\ \Gamma_x\in\mathbb{R}^{d\times d},\ \Gamma_x>0\\ \hat{\boldsymbol{\theta}}_{P,x}(\mathbf{x},t)&=\psi_x(\mathbf{x},t)\boldsymbol{\alpha}_x(\mathbf{x},t)-\Psi_x(\mathbf{x},t)\\ \dot{\hat{\boldsymbol{\theta}}}_{I,x}&=\varphi_x(\psi_x(\mathbf{x},t),\boldsymbol{\omega}_x,t)\boldsymbol{\alpha}_x(\mathbf{x},t)+\mathcal{R}_x(\mathbf{x},\hat{\boldsymbol{\theta}}_x,u_x(\mathbf{x},\hat{\boldsymbol{\theta}}_x,t),t),\end{aligned} \tag{35}$$

$$\begin{aligned}\hat{\boldsymbol{\theta}}_y(\mathbf{x},t)&=\Gamma_y(\hat{\boldsymbol{\theta}}_{P,y}(\mathbf{y},t)+\hat{\boldsymbol{\theta}}_{I,y}(t));\ \Gamma_y\in\mathbb{R}^{d\times d},\ \Gamma_y>0\\ \hat{\boldsymbol{\theta}}_{P,y}(\mathbf{y},t)&=\psi_y(\mathbf{y},t)\boldsymbol{\alpha}_y(\mathbf{y},t)-\Psi_y(\mathbf{y},t)\\ \dot{\hat{\boldsymbol{\theta}}}_{I,y}&=\varphi_y(\psi_y(\mathbf{y},t),\boldsymbol{\omega}_y,t)\boldsymbol{\alpha}_y(\mathbf{y},t)+\mathcal{R}_y(\mathbf{x},\hat{\boldsymbol{\theta}}_y,u_y(\mathbf{y},\hat{\boldsymbol{\theta}}_y,t),t),\end{aligned} \tag{36}$$

where $\mathcal{R}_x(\cdot)$, $\mathcal{R}_y(\cdot)$ are defined as in (18), and the functions $\Psi_x(\cdot)$, $\Psi_y(\cdot)$ will be specified later. Now we are ready to formulate the following result

Theorem 2 (Properties of the interconnected systems) *Let systems (30), (31) be given. Furthermore, suppose that the following conditions hold:*

1) The functions $\psi_x(\mathbf{x},t)$, $\psi_y(\mathbf{y},t)$ satisfy Assumption 1 for systems (30), (31) respectively;

2) The systems

$$\dot{\psi}_x=-\varphi_x(\psi_x,\boldsymbol{\omega}_x,t)+\zeta_x(t),\quad \dot{\psi}_y=-\varphi_y(\psi_y,\boldsymbol{\omega}_y,t)+\zeta_y(t) \tag{37}$$

satisfy Assumption 2 with corresponding mappings

$$\gamma_{x_{\infty,2}}(\psi_{x_0},\boldsymbol{\omega}_x,\|\zeta_x(t)\|_{2,[t_0,T]}),\quad \gamma_{y_{\infty,2}}(\psi_{y_0},\boldsymbol{\omega}_y,\|\zeta_y(t)\|_{2,[t_0,T]}),$$

3) The systems (37) have $L_2^1[t_0,\infty]\mapsto L_2^1[t_0,\infty]$ gains, that is

$$\begin{aligned}&\|\psi_x(\mathbf{x}(t),t)\|_{2,[t_0,T]}\le C_{\gamma_x}+\gamma_{x_{2,2}}(\|\zeta_x(t)\|_{2,[t_0,T]}),\\ &\|\psi_y(\mathbf{y}(t),t)\|_{2,[t_0,T]}\le C_{\gamma_y}+\gamma_{y_{2,2}}(\|\zeta_y(t)\|_{2,[t_0,T]}),\\ &\quad C_{\gamma_x},\ C_{\gamma_y}\in\mathbb{R}_+\gamma_{x_{2,2}},\ \gamma_{y_{2,2}}\in\mathcal{K}_\infty\end{aligned} \tag{38}$$

4) The functions $f_x(\mathbf{x}, \boldsymbol{\theta}_x, t)$, $f_y(\mathbf{y}, \boldsymbol{\theta}_y, t)$ satisfy Assumptions 3, 4 with corresponding constants D_x, D_{x_1}, D_y, D_{y_1} and functions $\boldsymbol{\alpha}_x(\mathbf{x}, t)$, $\boldsymbol{\alpha}_y(\mathbf{y}, t)$;

5) The functions $h_x(\mathbf{x}, \mathbf{y}, t)$, $h_y(\mathbf{x}, \mathbf{y}, t)$ satisfy the following inequalities:

$$\|h_x(\mathbf{x}, \mathbf{y}, t)\| \leq \beta_x \|\psi_x(\mathbf{x}, t)\|, \ \|h_y(\mathbf{x}, \mathbf{y}, t)\| \leq \beta_y \|\psi_y(\mathbf{y}, t)\|, \ \beta_x, \beta_y \in \mathbb{R}_+ \tag{39}$$

Finally, let the functions $\Psi_x(\mathbf{x}, t)$, $\Psi_y(\mathbf{y}, t)$ in (35), (36) satisfy Assumption 5 for systems (30), (31) respectively, and there exist functions $\rho_1(\cdot)$, $\rho_2(\cdot)$, $\rho_3(\cdot) > Id(\cdot) \in \mathcal{K}_\infty$ and constant $\bar{\Delta} \in \mathbb{R}_+$ such the following inequality holds:

$$\beta_y \circ \gamma_{y2,2} \circ \rho_1 \circ \left(\frac{D_y}{D_{y,1}} + 1 \right) \circ \rho_3 \circ \beta_x \circ \gamma_{x2,2} \circ \rho_2 \circ \left(\frac{D_x}{D_{x,1}} + 1 \right) (\Delta) < \Delta \tag{40}$$

for all $\Delta \geq \bar{\Delta}$. Then

C1) The interconnection (30), (31) with controls (32), (33) is forward-complete and trajectories $\mathbf{x}(t)$, $\mathbf{y}(t)$ are bounded

Furthermore,

C2) if properties H1, H3 hold for $f_x(\mathbf{x}, \boldsymbol{\theta}_x, t)$, $f_y(\mathbf{y}, \boldsymbol{\theta}_y, t)$, $h_x(\mathbf{x}, \mathbf{y}, t)$, $h_y(\mathbf{x}, \mathbf{y}, t)$, and also functions $\varphi_x(\psi_x, \boldsymbol{\omega}_x, t)$, $\varphi_y(\psi_y, \boldsymbol{\omega}_y, t)$, then

$$\lim_{t \to \infty} \psi_x(\mathbf{x}(t), t) = 0, \ \lim_{t \to \infty} \psi_y(\mathbf{y}(t), t) = 0 \tag{41}$$

Moreover,

C3) if property H2 holds for $f_x(\mathbf{x}, \boldsymbol{\theta}_x, t)$, $f_y(\mathbf{y}, \boldsymbol{\theta}_y, t)$, and the functions

$$\boldsymbol{\alpha}_x(\mathbf{x}, t), \ \partial\psi_x(\mathbf{x}, t)/\partial t, \ \boldsymbol{\alpha}_y(\mathbf{y}, t), \ \partial\psi_y(\mathbf{y}, t)/\partial t$$

are locally bounded with respect to $\mathbf{x}$, $\mathbf{y}$ uniformly in t, then

$$\begin{aligned} &\lim_{t \to \infty} f_x(\mathbf{x}(t), \boldsymbol{\theta}_x, t) - f_x(\mathbf{x}(t), \hat{\boldsymbol{\theta}}_x(t), t) = 0, \\ &\lim_{t \to \infty} f_y(\mathbf{y}(t), \boldsymbol{\theta}_y, t) - f_y(\mathbf{y}(t), \hat{\boldsymbol{\theta}}_y(t), t) = 0 \end{aligned} \tag{42}$$

Let us briefly comment on the conditions and assumptions of Theorem 2. Conditions 1), 2) specify restrictions on the goal functionals, similar to those of Theorem 1. Condition 3) is analogous to requirement to P3) in Theorem 1, condition 5) specifies uncertainties in the coupling functions $h_x(\cdot)$, $h_y(\cdot)$ in terms of their growth rates w.r.t. $\psi_x(\cdot)$, $\psi_y(\cdot)$. We observe here that this property is needed in order to characterize the L_2 norms of functions $h_x(\mathbf{x}(t), \mathbf{y}(t), t)$, $h_y(\mathbf{x}(t), \mathbf{y}(t), t)$ in terms of the L_2 norms of functions $\psi_x(\mathbf{x}(t), t)$, $\psi_y(\mathbf{y}(t), t)$. Therefore, it is possible to replace requirement (39) with the following set of conditions:

$$\begin{aligned} &\|h_x(\mathbf{x}(t), \mathbf{y}(t), t)\|_{2,[t_0,T]} \leq \beta_x \|\psi_x(\mathbf{x}(t), t)\|_{2,[t_0,T]} + C_x, \\ &\|h_y(\mathbf{x}(t), \mathbf{y}(t), t)\|_{2,[t_0,T]} \leq \beta_y \|\psi_y(\mathbf{y}(t), t)\|_{2,[t_0,T]} + C_y \end{aligned} \tag{43}$$

The replacement will allow us to extend results of Theorem 2 to interconnections of systems where the coupling functions do not depend explicitly on $\psi_x(\mathbf{x}(t), t)$, $\psi_y(\mathbf{y}(t), t)$. We illustrate this possibility later with an example.

Condition (40) is the small-gain condition with respect to the $L_2^1[t_0, T]$ norms for interconnection (30), (31) with control (32), (33). In the case that mappings $\gamma_{x_{2,2}}(\cdot)$, $\gamma_{y_{2,2}}(\cdot)$ in (37) are majorated by linear functions

$$\gamma_{x_{2,2}}(\Delta) \leq g_{x_{2,2}}\Delta,\ \gamma_{y_{2,2}}(\Delta) \leq g_{y_{2,2}}\Delta,\ \Delta \geq 0,$$

condition (40) reduces to the much simpler

$$\beta_y \beta_x g_{x_{2,2}} g_{y_{2,2}} \left(\frac{D_y}{D_{y,1}} + 1\right)\left(\frac{D_x}{D_{x,1}} + 1\right) < 1$$

Notice also that the mappings $\gamma_{x_{2,2}}(\cdot)$, $\gamma_{y_{2,2}}(\cdot)$ are defined by properties of the target dynamics (37), and, in principle, these can be made arbitrarily small. This eventually leads to the following conclusion: the smaller the L_2-gains of the target dynamics of systems $\mathcal{S}_1$, $\mathcal{S}_2$, the wider the class of nonlinearities (bounds for β_x, β_y, domains of D_x, $D_{1,x}$, D_y, $D_{1,y}$) which admit a solution to Problem 1.

Example Let us illustrate application of Theorem 2 to the problem of decentralized control of two coupled oscillators with nonlinear damping. Consider the following interconnected systems:

$$\left\{\begin{array}{ll} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = f_x(x_1, \theta_x) + k_1 y_1 + u_x, \end{array}\right. \quad \left\{\begin{array}{ll} \dot{y}_1 & = y_2 \\ \dot{y}_{22} & = f_y(y_1, \theta_y) + k_2 x_1 + u_y, \end{array}\right. \tag{44}$$

where k_1, $k_2 \in \mathbb{R}$ are uncertain parameters of coupling, functions $f(x_1, \theta_x)$, $f(y_1, \theta_y)$ stand for the nonlinear damping terms, and θ_x, θ_y are unknown parameters. For illustrative purpose we assume the following mathematical model for functions $f_x(\cdot)$, $f_y(\cdot)$ in (44):

$$\begin{aligned} f_x(x_1, \theta_x) &= \theta_x(x_1 - x_0) + 0.5\sin(\theta_x(x_1 - x_0)), \\ f_y(y_1, \theta_y) &= \theta_y(y_1 - y_0) + 0.6\sin(\theta_y(y_1 - y_0)) \end{aligned} \tag{45}$$

where x_0, y_0 are known. Let the control goal be to steer states $\mathbf{x}$ and $\mathbf{y}$ to the origin. Consider the following goal functions

$$\psi_x(\mathbf{x}, t) = x_1 + x_2,\ \psi_y(\mathbf{y}, t) = y_1 + y_2 \tag{46}$$

Taking into account equations (44) and (46) we can derive that

$$\dot{x}_1 = -x_1 + \psi_x(\mathbf{x}(t), t),\ \dot{y}_1 = -y_1 + \psi_y(\mathbf{y}, t) \tag{47}$$

This automatically implies that

$$\begin{aligned} \|x_1(t)\|_{\infty,[t_0,T]} &\leq \|x_1(t_0)\| + \|\psi_x(\mathbf{x}(t), t)\|_{\infty,[t_0,T]} \\ \|y_1(t)\|_{\infty,[t_0,T]} &\leq \|y_1(t_0)\| + \|\psi_y(\mathbf{y}(t), t)\|_{\infty,[t_0,T]} \end{aligned}$$

Hence, Assumption 1 is satisfied for chosen goal functions $\psi_x(\cdot)$ and $\psi_y(\cdot)$. Notice also that equalities (47) imply that

$$\begin{aligned} \|x_1(t)\|_{2,[t_0,T]} &\leq 2^{-1/2}\|x_1(t_0)\| + \|\psi_x(\mathbf{x}, t)\|_{2,[t_0,T]} \\ \|y_1(t)\|_{2,[t_0,T]} &\leq 2^{-1/2}\|y_1(t_0)\| + \|\psi_y(\mathbf{y}, t)\|_{2,[t_0,T]} \end{aligned} \tag{48}$$

Moreover, according to (47) limiting relations

$$\begin{aligned} \lim_{t\to\infty} \psi_x(\mathbf{x}(t),t) &= \lim_{t\to\infty} x_1(t)+x_2(t)=0, \\ \lim_{t\to\infty} \psi_y(\mathbf{y}(t),t) &= \lim_{t\to\infty} y_1(t)+y_2(t)=0 \end{aligned} \tag{49}$$

guarantee that

$$\lim_{t\to\infty} x_1(t)=0,\ \lim_{t\to\infty} x_2(t)=0,\ \lim_{t\to\infty} y_1(t)=0,\ \lim_{t\to\infty} y_2(t)=0$$

Hence, property (49) ensures asymptotic reaching of the control goal.

According to equations (32), (33) control functions

$$\begin{aligned} u_x &= -\lambda_x\psi_x - x_2 - f_x(x_1,\hat{\theta}_x) \\ u_y &= -\lambda_y\psi_y - y_2 - f_y(y_1,\hat{\theta}_y),\ \lambda_x,\ \lambda_y>0 \end{aligned} \tag{50}$$

transform system (44) into the following form

$$\begin{aligned} \dot{\psi}_x &= -\lambda_x\psi_x + f_x(x_1,\theta_x) - f_x(x_1,\hat{\theta}_x) + k_1 y_1 \\ \dot{\psi}_x &= -\lambda_x\psi_x + f_x(x_1,\theta_x) - f_x(x_1,\hat{\theta}_x) + k_2 x_1 \end{aligned} \tag{51}$$

Notice that systems

$$\dot{\psi}_x = -\lambda_x\psi_x + \xi_x(t),\ \dot{\psi}_y = -\lambda_y\psi_t + \xi_y(t)$$

satisfy Assumption 2 with

$$\gamma_{x2,2} = \frac{1}{\lambda_x}\|\psi_x(\mathbf{x}(t),t)\|_{2,[t_0,T]},\ \gamma_{y2,2} = \frac{1}{\lambda_y}\|\psi_y(\mathbf{y}(t),t)\|_{2,[t_0,T]}$$

respectively, and functions $f_x(\cdot)$, $f_y(\cdot)$ satisfy Assumptions 3, 4 with

$$\begin{aligned} &D_x=1.5,\ D_{x,1}=0.5,\ \alpha_x(\mathbf{x},t)=x_1-x_0, \\ &D_y=1.6,\ D_{y,1}=0.4,\ \alpha_y(\mathbf{y},t)=y_1-y_0 \end{aligned}$$

Hence conditions 1)-4) of Theorem 2 are satisfied. Furthermore, according to the remarks regarding condition 5) of the theorem, requirements (39) can be replaced with implicit constraints (43). These, however, according to (48) also hold with $\beta_x=k_1$, $\beta_y=k_2$.

Given that $\alpha_x(\mathbf{x},t)=x_1-x_0$, $\alpha_y(\mathbf{y},t)=y_1-y_0$, Assumption 5 will be satisfied for functions $\alpha_x(\mathbf{x},t)$, $\alpha_y(\mathbf{y},t)$ with $\Psi_x(\cdot)=0$, $\Psi_y(\cdot)=0$. Therefore, adaptation algorithms (35), (36) will have the following form:

$$\begin{aligned} \hat{\theta}_x &= \Gamma_x((x_1+x_2)(x_1-x_0)+\hat{\theta}_{x,I}), \\ \dot{\hat{\theta}}_{x,I} &= \lambda_x(x_1+x_2)(x_1-x_0)-(x_1+x_2)x_2 \\ \hat{\theta}_y &= \Gamma_y((y_1+y_2)(y_1-y_0)+\hat{\theta}_{y,I}), \\ \dot{\hat{\theta}}_{y,I} &= \lambda_y(y_1+y_2)(y_1-y_0)-(y_1+y_2)y_2 \end{aligned} \tag{52}$$

Hence, according to Theorem 2 boundedness of the solutions in the closed loop system (51), (52) is ensured upon the following condition

$$\frac{k_1k_2}{\lambda_x\lambda_y}\left(1+\frac{D_x}{D_{x,1}}\right)\left(1+\frac{D_y}{D_{y,1}}\right)<1 \Rightarrow k_1k_2<\frac{\lambda_x\lambda_y}{20} \tag{53}$$

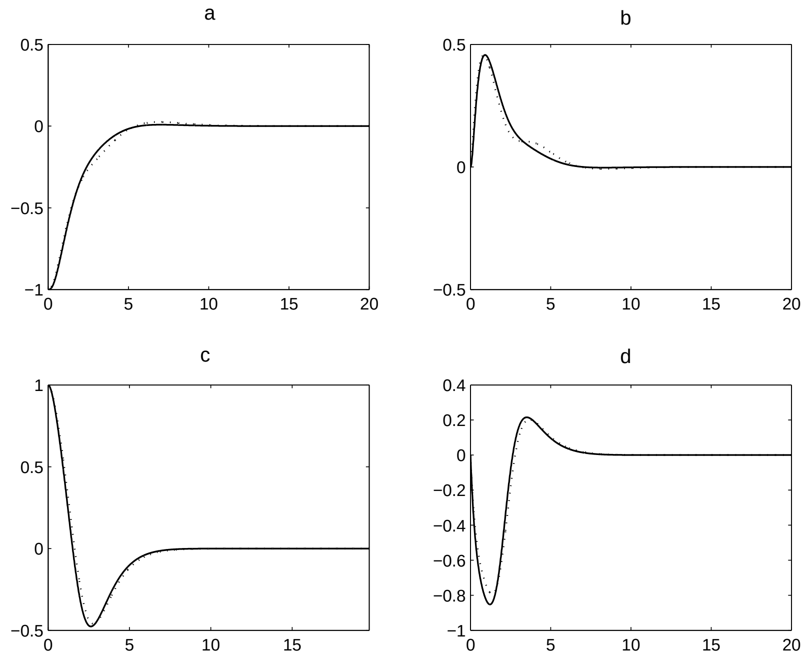

Figure 2: Plots of trajectories $x_1(t)$ (panel a), $x_2(t)$ (panel b), $y_1(t)$ (panel c), $y_2(t)$ (panel d) as functions of t in closed loop system (44), (50), (52). Dotted lines correspond to the case when $k_1 = k_2 = 0.4$, and solid lines stand for solutions obtained with the following values of coupling $k_1 = 1$, $k_2 = 0.1$

Moreover, given that properties H1– H3 hold for the chosen functions $\psi_x(\mathbf{x}, t)$, $\psi_y(\mathbf{y}, t)$, condition (53) guarantees that limiting relations (41), (42) hold.

Trajectories of the closed loop system (44), (50), (52) with the following values of parameters $\Gamma_x = \Gamma_y = 1$, $\lambda_x = \lambda_y = 2$, $x_0 = y_0 = 1$, $\theta_x = \theta_y = 1$ and initial conditions $x_1(0) = -1$, $x_2(0) = 0$, $y_1(0) = 1$, $y_2(0) = 0$, $\hat{\theta}_{x,I}(0) = -1$, $\hat{\theta}_{y,I}(0) = -2$ are provided in Fig. 2.

5 Conclusion

We provided new tools for the design and analysis of adaptive decentralized control schemes. Our method allows the desired dynamics to be Lyapunov unstable and the parametrization of the uncertainties to be nonlinear. The results are based on a formulation of the problem for adaptive control as a problem of regulation in functional spaces (in particular, $L_2^1[t_0, T]$ spaces) rather than of simply reaching of the control goal in $\mathbb{R}^n$. This allows us to introduce adaptation algorithms with new properties and apply a small-gain argument to establish applicability of these schemes to the problem of decentralized control.

In order to avoid unnecessary complications, state feedback was assumed in the main-loop controllers which transform original equation into the error coupled model. Extension of the results to output-feedback main loop controllers is a topic for future study.

6 Proofs of the theorems

6.1 Proof of Theorem 1

Let us first show that property P1) holds. Consider solutions of system (4), (10), (17), (18) passing through the point $\mathbf{x}(t_0)$, $\hat{\boldsymbol{\theta}}_I(t_0)$ for $t \in [t_0, T^*]$. Let us calculate the time-derivative of function $\hat{\boldsymbol{\theta}}(\mathbf{x}, t)$: $\dot{\hat{\boldsymbol{\theta}}}(\mathbf{x}, t) = \Gamma(\dot{\hat{\boldsymbol{\theta}}}_P + \dot{\hat{\boldsymbol{\theta}}}_I) = \Gamma(\dot{\psi}\boldsymbol{\alpha}(\mathbf{x}, t) + \psi\dot{\boldsymbol{\alpha}}(\mathbf{x}, t) - \dot{\Psi}(\mathbf{x}, t) + \dot{\hat{\boldsymbol{\theta}}}_I)$. Notice that

$$\begin{aligned}
&\psi\dot{\boldsymbol{\alpha}}(\mathbf{x}, t) - \dot{\Psi}(\mathbf{x}, t) + \dot{\hat{\boldsymbol{\theta}}}_I = \psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x}, t)}{\partial \mathbf{x}_1}\dot{\mathbf{x}}_1 + \psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x})}{\partial \mathbf{x}_2}\dot{\mathbf{x}}_2 + \\
&\psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x}, t)}{\partial t} - \frac{\partial \Psi(\mathbf{x}, t)}{\partial \mathbf{x}_1}\dot{\mathbf{x}}_1 - \frac{\partial \Psi(\mathbf{x}, t)}{\partial \mathbf{x}_2}\dot{\mathbf{x}}_2 - \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} + \dot{\hat{\boldsymbol{\theta}}}_I
\end{aligned} \tag{54}$$

According to Assumption 5, $\frac{\partial \Psi(\mathbf{x},t)}{\partial \mathbf{x}_2} = \psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial \mathbf{x}_2}$. Then taking into account (54), we obtain

$$\begin{aligned}
&\psi\dot{\boldsymbol{\alpha}}(\mathbf{x}, t) - \dot{\Psi}(\mathbf{x}, t) + \dot{\hat{\boldsymbol{\theta}}}_I = \left(\psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x}, t)}{\partial \mathbf{x}_1} - \frac{\partial \Psi}{\partial \mathbf{x}_1}\right)\dot{\mathbf{x}}_1 \\
&+ \psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x}, t)}{\partial t} - \frac{\Psi(\mathbf{x}, t)}{\partial t}
\end{aligned} \tag{55}$$

Notice that according to the proposed notation we can rewrite the term $\left(\psi(\mathbf{x}, t)\frac{\partial \boldsymbol{\alpha}(\mathbf{x},t)}{\partial \mathbf{x}_1} - \frac{\partial \Psi}{\partial \mathbf{x}_1}\right)\dot{\mathbf{x}}_1$ in the following form: $\psi(\mathbf{x}, t)L_{\mathbf{f}_1}\boldsymbol{\alpha}(\mathbf{x}, t) - L_{\mathbf{f}_1}\Psi(\mathbf{x}, t) + (\psi(\mathbf{x}, t)L_{\mathbf{g}_1}\boldsymbol{\alpha}(\mathbf{x}, t) - L_{\mathbf{g}_1}\Psi(\mathbf{x}, t))\, u(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$. Hence, it follows from (17) and (55) that $\psi\dot{\boldsymbol{\alpha}}(\mathbf{x}, t) - \dot{\Psi}(\mathbf{x}, t) + \dot{\hat{\boldsymbol{\theta}}}_I = \varphi(\psi)\boldsymbol{\alpha}(\mathbf{x}, t)$. Therefore, the derivative $\dot{\hat{\boldsymbol{\theta}}}(\mathbf{x}, t)$ can be written in the following way:

$$\dot{\hat{\boldsymbol{\theta}}} = \Gamma(\dot{\psi} + \varphi(\psi))\boldsymbol{\alpha}(\mathbf{x}, t) \tag{56}$$

Asymptotic properties of nonlinear parameterized control systems with adaptation algorithm (56) under assumption of Lyapunov stability of the target dynamics were investigated in [22]. In the present contribution we aim to provide characterizations of the closed loop system in terms of functional mappings between functions $\psi(\mathbf{x}(t), t)$, $\varepsilon(t)$, and $f(\mathbf{x}(t), \boldsymbol{\theta}, t) - f(\mathbf{x}(t), \hat{\boldsymbol{\theta}}(t), t)$ and without requiring Lyapunov stability of the target dynamics (11).

For this purpose consider the following positive-definite function:

$$V_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}, t) = \frac{1}{2}\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{\Gamma^{-1}} + \frac{D}{4D_1^2}\int_t^\infty \varepsilon^2(\tau)d\tau \tag{57}$$

Its time-derivative according to equations (56) can be obtained as follows:

$$\dot{V}_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}, t) = (\varphi(\psi) + \dot{\psi})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\boldsymbol{\alpha}(\mathbf{x}, t) - \frac{D}{4D_1^2}\varepsilon^2(t) \tag{58}$$

Hence using Assumptions 3, 4 and equality (10) we can estimate the derivative $\dot{V}_{\hat{\boldsymbol{\theta}}}$ as follows:

$$\begin{aligned}
&\dot{V}_{\hat{\boldsymbol{\theta}}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}, t) \leq -(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t) + \varepsilon(t))(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\boldsymbol{\alpha}(\mathbf{x}, t) - \frac{D}{4D_1^2}\varepsilon^2(t) \\
&\leq -\frac{1}{D}(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t))^2 + \frac{1}{D_1}|\varepsilon(t)||f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| \\
&-\frac{D}{4D_1^2}\varepsilon^2(t) \leq -\frac{1}{D}\left(|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)| - \frac{D}{2D_1}\varepsilon(t)\right)^2 \leq 0
\end{aligned} \tag{59}$$

It follows immediately from (59), (57) that

$$\|\hat{\boldsymbol{\theta}}(t)-\boldsymbol{\theta}\|^2_{\Gamma^{-1}} \leq \|\hat{\boldsymbol{\theta}}(t_0)-\boldsymbol{\theta}\|^2_{\Gamma^{-1}} + \frac{D}{2D_1^2}\|\varepsilon(t)\|^2_{2,[t_0,\infty]} \tag{60}$$

In particular, for $t \in [t_0, T^*]$ we can derive from (57) that $\|\hat{\boldsymbol{\theta}}(t)-\boldsymbol{\theta}\|^2_{\Gamma^{-1}} \leq \|\hat{\boldsymbol{\theta}}(t_0)-\boldsymbol{\theta}\|^2_{\Gamma^{-1}} + \frac{D}{2D_1^2}\|\varepsilon(t)\|^2_{2,[t_0,T^*]}$. Therefore $\hat{\boldsymbol{\theta}}(t) \in L^2_\infty[t_0, T^*]$. Furthermore $|f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)| - \frac{D}{2D_1}\varepsilon(t) \in L^1_2[t_0,T^*]$. In particular

$$\begin{aligned}&\left\| |f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)| - \frac{D}{2D_1}\varepsilon(t)\right\|^2_{2,[t_0,T^*]} \leq \\ &\frac{D}{2}\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}(t_0)\|^2_{\Gamma^{-1}} + \frac{D^2}{4D_1^2}\|\varepsilon(t)\|^2_{2,[t_0,T^*]}\end{aligned} \tag{61}$$

Hence $f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t) \in L^1_2[t_0,T^*]$ as a sum of two functions from $L^1_2[t_0,T^*]$. In order to estimate the upper bound of the norm $\|f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)\|_{2,[t_0,T^*]}$ from (61) we use the Minkowski inequality:

$$\begin{aligned}&\left\| f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)| - \frac{D}{2D_1}\varepsilon(t)\right\|_{2,[t_0,T^*]} \leq \\ &\left(\frac{D}{2}\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}(t_0)\|^2_{\Gamma^{-1}}\right)^{0.5} + \frac{D}{2D_1}\|\varepsilon(t)\|_{2,[t_0,T^*]}\end{aligned}$$

and then apply the triangle inequality to the functions from $L^1_2[t_0,T^*]$:

$$\begin{aligned}&\|f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)\|_{2,[t_0,T^*]} \leq \\ &\left\| f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t) - \frac{D}{2D_1}\varepsilon(t)\right\|_{2,[t_0,T^*]} + \\ &\frac{D}{2D_1}\|\varepsilon(t)\|_{2,[t_0,T^*]} \leq \left(\frac{D}{2}\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}(t_0)\|^2_{\Gamma^{-1}}\right)^{0.5} + \frac{D}{D_1}\|\varepsilon(t)\|_{2,[t_0,T^*]}\end{aligned} \tag{62}$$

Therefore, property P1) is proven.

Let us prove property P2). In order to do this we have to check first if the solutions of the closed loop system are defined for all $t \in \mathbb{R}_+$, i.e. they do not go to infinity in finite time. We prove this by a contradiction argument. Indeed, let there exists time instant t_s such that $\|\mathbf{x}(t_s)\| = \infty$. It follows from P1), however, that $f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t) \in L^1_2[t_0,t_s]$. Furthermore, according to (62) the norm $\|f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)-f(\mathbf{x}(t),\boldsymbol{\theta},t)\|_{2,[t_0,t_s]}$ can be bounded from above by a continuous function of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}(t_0)$, Γ, and $\|\varepsilon(t)\|_{2,[t_0,\infty]}$. Let us denote this bound by symbol D_f. Notice that D_f does not depend on t_s. Consider system (10) for $t \in [t_0, t_s]$: $\dot{\psi} = f(\mathbf{x},\boldsymbol{\theta},t) - f(\mathbf{x},\hat{\boldsymbol{\theta}},t) - \varphi(\psi,\boldsymbol{\omega},t) + \varepsilon(t)$. Given that both $f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t), \varepsilon(t) \in L^1_2[t_0,t_s]$ and taking into account Assumption 2, we automatically obtain that $\psi(\mathbf{x}(t),t) \in L^1_\infty[t_0,t_s]$. In particular, using the triangle inequality and the fact that the function $\gamma_{\infty,2}(\psi(\mathbf{x}_0,t_0),\boldsymbol{\omega},M)$ in Assumption 2 is non-decreasing in M, we can estimate the norm $\|\psi(\mathbf{x}(t),t)\|_{\infty,[t_0,t_s]}$ as follows:

$$\|\psi(\mathbf{x}(t),t)\|_{\infty,[t_0,t_s]} \leq \gamma_{\infty,2}\left(\psi(\mathbf{x}_0,t_0),\boldsymbol{\omega}, D_f + \|\varepsilon(t)\|^2_{2,[t_0,\infty]}\right) \tag{63}$$

According to Assumption 1 the following inequality holds:

$$\|\mathbf{x}(t)\|_{\infty,[t_0,t_s]} \leq \tilde{\gamma}\left(\mathbf{x}_0, \boldsymbol{\theta}, \gamma_{\infty,2}\left(\psi(\mathbf{x}_0,t_0), \boldsymbol{\omega}, D_f + \|\varepsilon(t)\|^2_{2,[t_0,\infty]}\right)\right) \tag{64}$$

Given that a superposition of locally bounded functions is locally bounded, we conclude that $\|\mathbf{x}(t)\|_{\infty[t_0,t_s]}$ is bounded. This, however, contradicts to the previous claim that $\|\mathbf{x}(t_s)\| = \infty$. Taking into account inequality (60) we can derive that both $\hat{\boldsymbol{\theta}}(\mathbf{x}(t),t)$ and $\hat{\boldsymbol{\theta}}_I(t)$ are bounded for every $t \in \mathbb{R}_+$. Moreover, according to (63), (64), (60) these bounds are themselves locally bounded functions of initial conditions and parameters. Therefore, $\mathbf{x}(t) \in L^n_\infty[t_0,\infty]$, $\hat{\boldsymbol{\theta}}(\mathbf{x}(t),t) \in L^d_\infty[t_0,\infty]$. Inequality (21) follows immediately from (62), (12), and the triangle inequality. Property P2) is proven.

Let us show that P3) holds. It is assumed that system (11) has $L^1_2[t_0,\infty] \mapsto L^1_p[t_0,\infty]$, $p>1$ gain. In addition, we have just shown that $f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t), \varepsilon(t) \in L_2[t_0,\infty]$. Hence, taking into account equation (10) we conclude that $\psi(\mathbf{x}(t),t) \in L^1_p[t_0,\infty]$, $p>1$. On the other hand, given that $f(\mathbf{x},\hat{\boldsymbol{\theta}},t)$, $\varphi(\psi,\boldsymbol{\omega},t)$ are locally bounded with respect to their first two arguments uniformly in t, and that $\mathbf{x}(t) \in L^n_\infty[t_0,\infty]$,$\psi(\mathbf{x}(t),t) \in L^1_\infty[t_0,\infty]$, $\hat{\boldsymbol{\theta}}(t) \in L^d_\infty[t_0,\infty]$, $\boldsymbol{\theta} \in \Omega_\theta$, the signal $\varphi(\psi(\mathbf{x}(t),t),\boldsymbol{\omega},t) + f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t)$ is bounded. Then $\varepsilon(t) \in L^1_\infty[t_0,\infty]$ implies that $\dot{\psi}$ is bounded, and P3) is guaranteed by Barbalat's lemma.

To complete the proof of the theorem (property P4) consider the time-derivative of function $f(\mathbf{x},\hat{\boldsymbol{\theta}},t)$:

$$\frac{d}{dt}f(\mathbf{x},\hat{\boldsymbol{\theta}},t) = L_{\mathbf{f}(\mathbf{x},\boldsymbol{\theta})+\mathbf{g}(\mathbf{x})u(\mathbf{x},\hat{\boldsymbol{\theta}},t)} f(\mathbf{x},\hat{\boldsymbol{\theta}},t) +$$
$$\frac{\partial f(\mathbf{x},\hat{\boldsymbol{\theta}},t)}{\partial \hat{\boldsymbol{\theta}}}\Gamma(\varphi(\psi,\boldsymbol{\omega},t)+\dot{\psi})\boldsymbol{\alpha}(\mathbf{x},t) + \frac{\partial f(\mathbf{x},\hat{\boldsymbol{\theta}},t)}{\partial t}$$

Taking into account that the function $f(\mathbf{x},\boldsymbol{\theta},t)$ is continuously differentiable in $\mathbf{x}$, $\boldsymbol{\theta}$; the derivative $\partial f(\mathbf{x},\boldsymbol{\theta},t)/\partial t$ is locally bounded with respect to $\mathbf{x}$, $\boldsymbol{\theta}$ uniformly in t; functions $\boldsymbol{\alpha}(\mathbf{x},t)$, $\partial\psi(\mathbf{x},t)/\partial t$ are locally bounded with respect to $\mathbf{x}$ uniformly in t, then $d/dt(f(\mathbf{x},\boldsymbol{\theta},t) - f(\mathbf{x},\hat{\boldsymbol{\theta}},t))$ is bounded. Then given that $f(\mathbf{x}(t),\boldsymbol{\theta},t) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(t),t) \in L^1_2[t_0,\infty]$ by applying Barbalat's lemma we conclude that $f(\mathbf{x},\boldsymbol{\theta},\tau) - f(\mathbf{x},\hat{\boldsymbol{\theta}},\tau) \to 0$ as $t \to \infty$. The theorem is proven.

6.2 Proof of Theorem 2

Let us denote

$$\Delta f_x[t_0,T] = \|f_x(\mathbf{x},\boldsymbol{\theta}_x,t) - f_x(\mathbf{x},\hat{\boldsymbol{\theta}}_x,t)\|_{2,[t_0,T]},$$
$$\Delta f_y[t_0,T] = \|f_x(\mathbf{y},\boldsymbol{\theta}_y,t) - f_y(\mathbf{y},\hat{\boldsymbol{\theta}}_y,t)\|_{2,[t_0,T]}.$$

As follows from Theorem 1 the following inequalities hold

$$\Delta f_x[t_0,T] \leq C_x + \frac{D_x}{D_{1,x}}\|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]} \tag{65}$$

$$\Delta f_y[t_0,T] \leq C_y + \frac{D_y}{D_{1,y}}\|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}, \tag{66}$$

where C_x, C_y are some constants, independent of T. Taking estimates (65), (66) into account we obtain the following estimates:

$$\begin{aligned}
&\Delta f_x[t_0,T] + \|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]} \leq \\
&C_x + \left(\frac{D_x}{D_{1,x}}+1\right)\|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\Delta f_y[t_0,T] + \|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]} \leq \\
&C_y + \left(\frac{D_y}{D_{1,y}}+1\right)\|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]},
\end{aligned} \tag{68}$$

The proof of the theorem would be complete if we show that the $L_2^1[t_0,T]$ norms of $h_x(\mathbf{x}(t),\mathbf{y}(t),t)$, $h_y(\mathbf{x}(t),\mathbf{y}(t),t)$ are globally bounded uniformly in T. Let us show that this is indeed the case. Using the widely known generalized triangular inequality [10]

$$\gamma(a+b) \leq \gamma((\rho+Id)(a)) + \gamma((\rho+Id)\circ\rho^{-1}(b)), \ a,b\in\mathbb{R}_+, \ \gamma,\rho\in\mathcal{K}_\infty,$$

equations (67), (68) and also property (39), we conclude that

$$\begin{aligned}
&\|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]} \leq \\
&\beta_y\cdot\gamma_{y_{2,2}}\circ\rho_1\left(\left(\frac{D_y}{D_{1,y}}+1\right)\|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}\right) + C_{y,1} \\
&\|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]} \leq \\
&\beta_x\cdot\gamma_{x_{2,2}}\circ\rho_2\left(\left(\frac{D_x}{D_{1,x}}+1\right)\|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}\right) + C_{x,1}
\end{aligned} \tag{69}$$

where $\rho_1(\cdot)$, $\rho_2(\cdot)\in\mathcal{K}_\infty$, $\rho_1(\cdot),\rho_2(\cdot) > Id(\cdot)$. Then, according to (69), the existence of $\rho_3(\cdot)\in\mathcal{K}_\infty \geq Id(\cdot)$, satisfying inequality

$$\beta_y\circ\gamma_{y_{2,2}}\circ\rho_1\circ\left(\frac{D_y}{D_{y,1}}+1\right)\circ\rho_3\circ\beta_x\circ\gamma_{x_{2,2}}\circ\rho_2\circ\left(\frac{D_x}{D_{x,1}}+1\right)(\Delta) < \Delta \ \forall \ \Delta \geq \bar{\Delta}$$

for some $\bar{\Delta}\in\mathbb{R}_+$ ensures that the norms

$$\|h_y(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}, \ \|h_x(\mathbf{x}(t),\mathbf{y}(t),t)\|_{2,[t_0,T]}$$

are globally uniformly bounded in T. The rest of the proof follows from Theorem 1. The theorem is proven.

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